2-MARKS
Unit I Probability and Random Variable

1.. Define Random Variable (RV).

A random variable is a function X: S$\rightarrow$R that assigns a real number X(S) to every element $s \in S$, where S is the sample space corresponding to a random experiment E.

Ex: Consider an experiment of tossing an unbiased coin twice. The outcomes of the experiment are HH, HT, TH,TT.let X denote the number of heads turning up. Then X has the values 2,1,1,0. Here X is a random variable which assigns a real number to every outcome of a random experiment.

2. Define Discrete Random Variable.

If X is a random variable which can take a finite number or countably infinite number of pvalues, X is called a discrete RV.

Ex. Let X represent the sum of the numbers on the 2 dice, when two dice are trown.

3. Define Continuous Random Variable.

If X is a random variable which can take all values (i.e., infinite number of values) in an interval, then X is called a continuous RV.

Ex. The time taken by a lady who speaks over a telephone.

4. Define One-dimensional Random Variables.

If a random variable X takes on single value corresponding to each outcome of the experiment, then the random variable is called one-dimensional random variables.it is also called as scalar valued RVs.

Ex:

In coin tossing experiment, if we assume the random variable to be appearance of tail, then the sample space is{H,T} and the random variable is{1,0}.which is an one-dimensional random variables.

5. State the Properties of expectation.

If X and Y are random variables and a,b are constants, then

1.E(a)=a

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(a)= \sum_{i=1}^{n} ap_i = a\sum_{i=1}^{n} p_i = a(1) \ (\because \sum_{i=1}^{n} p_i = 1)$$

$$E(a) = a$$

2.E(aX)=aE(X)

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(aX)= \sum_{i=1}^{n} ax_i p_i = a\sum_{i=1}^{n} x_i p_i = aE(X)$$

3.E(aX+b)=aE(X)+b

Proof:

$$E(X)=\sum_{i=1}^{n} x_i p_i$$

$$E(aX+b)=\sum_{i=1}^{n}(ax_i+b)p_i=\sum_{i=1}^{n}(ax_i)p_i+\sum_{i=1}^{n}bp_i=a\sum_{i=1}^{n}x_ip_i+b\sum_{i=1}^{n}p_i$$

$$E(aX+b)=a\,E(X)+b \quad \{\because \sum_{i=1}^{n}p_i=1\}$$

4. $E(X+Y)=E(X)+E(Y)$
5. $E(XY)=E(X).E(Y)$, if X and Y are random variables.
6. $E(X-\overline{X})=E(X)-\overline{X}=\overline{X}-\overline{X}=0$

6. A RV X has the following probability function

Values of X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

1) Determine the value of a.
2) Find P(X<3), P(X$\geq$3), P(0<X<5).

Solution:

1) We know that $\sum_{x} P(x)=1$

a+3a+5a+7a+9a+11a+13a+15a+17a=1
81a = 1
a = 1/81

2) P(X<3) = P(X=0) + P(X=1) +P(X=2)
= a+3a+5a
=9a =9/81=1/9
P(X$\geq$3) = 1- P(X<3) =1-1/9 = 8/9
P(0<X<5) = P(X=1) + P(X=2) + P(X=3) +P(X=4)
= 3a+5a+7a +9a =24a= 24/81

7. If X is a continuous RV whose PDF is given by

$$f(x)=\begin{cases} c(4x-2x^2), 0<x<2 \\ 0, \quad otherwise \end{cases}$$

Find c.and mean of X.
Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_0^2 c(4x-2x^2)dx=1$$

c = 3/8

$$E(X)=\int_{-\infty}^{\infty} xf(x)dx=\int_0^2 \frac{3}{8}x(4x-2x^2)dx=\frac{8}{3}$$

8. A continuous RV X that can assume any value between x = 2 and x = 5 has a density function given by $f(x)=k(1+x)$. Fnd *P(X<4)*.
Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_2^5 k(1+x)dx=1$$

k = 2/27

$$P(X<4)=\int_2^4 \frac{2}{27}(1+x)dx=\frac{16}{27}$$

9. A RV X has the density function

$$f(x) = \begin{cases} k\dfrac{1}{1+x^2}, -\infty < x < \infty \\ 0, \quad otherwise \end{cases}$$ **. Find k .**

Solution:

We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} k\frac{1}{1+x^2}dx = 1$$

$$k(\tan^{-1} x)_{-\infty}^{\infty} = 1$$

$$k\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1$$

$$\therefore k = \frac{1}{\pi}$$

10. If the p.d.f of a RV .X is given by $f(x) = \begin{cases} \frac{1}{4}, -2 < X < 2 \\ 0, elsewhere \end{cases}$ **.Find** $P[|X| > 1]$.

Answer:

$$P[|X| > 1] = 1 - P[|X| < 1] = 1 - \int_{-1}^{1}\frac{1}{4}dx = 1 - \frac{1}{4}[1+1] = 1 - \frac{1}{2} = \frac{1}{2}$$

11. If the pdf of a RV X is $f(x) = \frac{x}{2}$ **in** $0 \le x \le 2$,find $P[X > 1.5 / X > 1]$

Answer:

$$P[X > 1.5 / X > 1] = \frac{p[X > 1.5]}{P[X > 1]} = \frac{\int_{1.5}^{2}\frac{x}{2}dx}{\int_{1}^{2}\frac{x}{2}dx} = \frac{4-2.25}{4-1} = 0.5833$$

12. Determine the Binomial distribution whose mean is 9 and whose SD is 3/2

Ans : np = 9 and npq = 9/4 $\therefore$ q = $\frac{npq}{np} = \frac{1}{4}$

$\Rightarrow$ p = 1- q = $\frac{3}{4}$. np = 9 $\Rightarrow$ n = $9\left(\frac{4}{3}\right)$=12.

P [x = r] = $12C_r\left(\frac{3}{4}\right)^r\left(\frac{1}{4}\right)^{12-r}$, r = 0,1,2,…,12.

13. Find the M.G.F of a Binomial distribution

$$M_x(t) = \sum_{i=0}^{n} e^{tx}\,{}_nC_x p^x q^{n-x} = \sum_{x=0}^{n} {}_nC_r (pe^t)^x q^{n-x} = (q + pe^t)^n$$

14. The mean and variance of the Binomial distribution are 4 and 3 respectively. Find P(X=0).

Ans :

mean = np = 4, Variance = npq = 3

$q = \frac{3}{4}$, $p = 1 - \frac{3}{4} = \frac{1}{4}$, np = 4 $\Rightarrow$ n = 16

P (X=0) $= {}_nC_0 p^0 q^{n-0} = 16C_0 p^0 q^{16-0} = \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{16} = \left(\frac{3}{4}\right)^{16}$

15. For a Binomial distribution mean is 6 and standard deviation is $\sqrt{2}$. Find the first two terms of the distribution.

Ans : Given np = 6, npq $= \left(\sqrt{2}\right)^2 = 2$

$q = \frac{2}{6} = \frac{1}{3}$, $p = 1 - q = \frac{2}{3}$, np = 6 $\Rightarrow n\left(\frac{2}{3}\right) = 6 \Rightarrow n = 9$

P (X= 0) = ${}_nC_0 p^0 q^{n-0} = 9C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{9-0} = \left(\frac{1}{3}\right)^9$

P (X=1) = ${}_nC_1 p^1 q^{n-1} = 9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8 = 6\left(\frac{1}{3}\right)^8$

16. The mean and variance of a binomial variate are 4 and $\frac{4}{3}$ respectively, find $P[X \geq 1]$.

Ans : np = 4, npq = $\frac{4}{3} \Rightarrow q = \frac{1}{3}, p = \frac{2}{3}$

$P[X \geq 1]$= 1- P[X<1] = 1 – P[X = 0] =1-$\left(\frac{1}{3}\right)^6$ = 0.9986

17. For a R.V X, $M_x(t) = \frac{1}{81}(e^t + 2)^4$. Find $P(X \leq 2)$.

Sol: Given $M_x(t) = \frac{1}{81}(e^t + 2)^4 = \left(\frac{e^t}{3} + \frac{2}{3}\right)^4$ ------------------- (1)

For Binomial Distribution, $M_x(t) = (q + pe^t)^n$. ------------------------- (2)

Comparing (1)&(2),

$\therefore n = 4,\ q = \frac{2}{3},\ p = \frac{1}{3}.$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 4C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 + 4C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 + 4C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)$$

$$= \frac{1}{81}(16 + 32 + 24) = \frac{72}{81} = 0.8889.$$

18. If a R.V X takes the values -1,0,1 with equal probability find the M.G.F of X.

Sol: P[X= -1]=1/3 , P[X=0]=1/3, P[X=1]=1/3

$$M_x(t) = \sum_x e^{tx} P(X = x) = \frac{1}{3}e^{-t} + \frac{1}{3} + \frac{1}{3}e^t = \frac{1}{3}(1 + e^t + e^{-t}).$$

19. A die is thrown 3 times. If getting a 6 is considered as success find the probability of atleast 2 success.

Sol: $p = \frac{1}{6}, q = \frac{5}{6} n = 3.$

P(at least 2 success)=P(X$\geq$2) = P(X=2)+P(X=3)

$$= 3C_2\left(\frac{1}{6}\right)^2 \frac{5}{6} + 3C_3\left(\frac{1}{6}\right)^3 = \frac{2}{27}.$$

20. Find p for a Binimial variate X if n=6,and 9P(X=4)=P(X=2).

Sol: $9P(X=4)=P(X=2)\Rightarrow 9\left({}_6C_4 p^4 q^2\right)={}_6C_2 p^2 q^4$

$$\Rightarrow 9p^2=q^2=(1-p)^2 \;\therefore 8p^2+2p-1=0$$

$$\therefore p=\frac{1}{4}\left(\because p\neq -\frac{1}{2}\right)$$

21. Comment on the following

"The mean of a BD is 3 and variance is 4"

For B.D, Variance< mean

$\therefore$ The given statement is wrongs

22. Define poisson distribution

A discrete RV X is said to follow Poisson Distribution with parameter λ if its probability mass function is p(x) = $\frac{e^{-\lambda}\lambda^{x}}{x!}$, x = 0, 1,2, …… ∞

23. If X is a Poisson variate such that P(X=2)=9P (X = 4) + 90 P(X=6), find the variance

Ans : P [X = x] = $\frac{e^{-\lambda}\lambda^{x}}{x!}$

Given P(X=2)=9P (X = 4) + 90 P(X=6)

$$\therefore \frac{e^{-\lambda}\lambda^2}{2!}=9\frac{e^{-\lambda}\lambda^4}{4!}+90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\Rightarrow \frac{1}{2}=\frac{9}{24}\lambda^2+\frac{90}{720}\lambda^4 \Rightarrow \lambda^4+3\lambda^2-4=0$$

$$\Rightarrow (\lambda^2+4)(\lambda^2-1)=0$$

$$\Rightarrow \lambda^2=-4 \quad \text{or } \lambda^2=1$$

hence $\lambda\ =1[\because \lambda^2\neq -4]$ Variance=1.

24. It is known that 5% of the books bound at a certain bindery have defective bindings. find the probability that 2 of 100 books bound by this bindery will have defective bindings.

Ans : Let X denote the number of defective bindings.

p = $\frac{5}{100}$ n = 100 $\therefore \lambda=np=5$

P [X =2] = $\frac{e^{-\lambda}\lambda^2}{2!}=\frac{e^{-5}(25)}{2}$ = 0.084

25. Find λ, if X follows Poisson Distribution such that P(X=2)=3P(X=3).

Sol: P(X=2)=3P(X=3) $\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!}=\frac{3e^{-\lambda}\lambda^3}{3!}\Rightarrow \frac{1}{2}=\frac{3\lambda}{6}\Rightarrow \lambda=1.$

26. If X is a Poisson variate such that $P(X=1)=\frac{3}{10}$ and $P(X=2)=\frac{1}{5}$.

Find $P(X=0)$ and $P(X=3)$.

Sol: $P(X=1)=\frac{3}{10} \quad \Rightarrow \frac{e^{-\lambda}\lambda}{1}=\frac{3}{10}$ ………..(1)

$P(X=2)=\frac{1}{5} \quad \Rightarrow \frac{e^{-\lambda}\lambda^2}{2}=\frac{1}{5}$ ……………..(2)

$$\frac{(2)}{(1)} \Rightarrow \frac{\lambda}{2} = \frac{10}{15} \Rightarrow \lambda = \frac{4}{3} \qquad \therefore P(X=0) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^0}{0!} = 0.2636$$

$$\therefore P(X=3) = \frac{e^{-\frac{4}{3}}\left(\frac{4}{3}\right)^3}{3!}$$

27. For a Poisson Variate X , $E(X^2) = 6$ .What is E(X).

Sol: $\lambda^2 + \lambda = 6 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = 2, -3.$

But $\lambda > 0 \therefore \lambda = 2$ Hence $E(X) = \lambda = 2$

28. A Certain Blood Group type can be find only in 0.05% of the people. If the population of a randomly selected group is 3000.What is the Probability that atleast a people in the group have this rare blood group.

Sol: p=0.05% =0.0005 n=3000 $\therefore \lambda = np = 1.5$

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - e^{-1.5}\left[1 + \frac{1.5}{1}\right] = 0.4422.$$

29. If X is a poisson variate with mean λ show that E[X^2] = λ E[X+1]

E[X^2] = $\lambda^2 + \lambda$

E (X+1) = E[X] + 1 $\quad \therefore E[X^2] = \lambda E[X+1]$

30. Find the M.G.F of Poisson Distribution.

Ans :

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

31. A discrete RV X has M.G.F $M_x(t) = e^{2(e^t-1)}$. Find E (X), var (X) and P(X=0)

Ans : $M_X(t) = e^{2(e^t-1)} \Rightarrow$ X follows Poisson Distribution $\therefore \lambda = 2$

Mean = E (X) = $\lambda = 2$ Var (X) = $\lambda = 2$

$$P[X=0] = \frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-2}2^0}{0!} = e^{-2}$$

32. If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6th attempt?

Ans : Given p = 0.5 q = 0.5

By Geometric distribution

P [X =x] = q^x p, x = 0,1,2………

since the target is destroyed on 6th attempt x = 5

$\therefore$ Required probability = q^x p = $(0.5)^6$ = 0.0157

33. Find the M.G.F of Geometric distribution

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} q^x p = p\sum_{x=0}^{\infty}(qe^t)^x$$

$$= p[1-qe^t]^{-1} = \frac{p}{1-qe^t}$$

34. Find the mean and variance of the distribution P[X=x]=2^{-x}, x=1,2,3….

Solution:

$$P[X=x]= \frac{1}{2^x} = \left(\frac{1}{2}\right)^{x-1}\frac{1}{2}, x = 1,2,3.....$$

$$\therefore p = \frac{1}{2} \ and \ \ q = \frac{1}{2}$$

$$Mean = \frac{q}{p} = 1; \ Variance = \frac{q}{p^2} = 2$$

35. Find the expected value and the variance of the number of times one must throw a die until the outcome 1 has occurred 4 times.

Solution:

X follows the negative bionomial distribution with parameter r = 4 and p=1/6

E(X)= mean = r P = rqQ = r (1-p) (1/p)= 20. (p=1/Q and q=P/Q)

Variance = rPQ = r(1-p)/p^2 = 120.

36. If a boy is throwing stones at a target, what is the probability that his 10^{th} throw is his 5^{th} hit, if the probability of hitting the target at any trial is ½?

Solution:

Since 10^{th} throw should result in the 5^{th} successes ,the first 9 throws ought to have resulted in 4 successes and 5 faliures.

n = 5,r = 5, $p = \frac{1}{2} = q$

$\therefore$ Required probability = P(X=5)= (5+5-1)C_5 $(1/2)^5$ $(1/2)^5$

=9C_4 $(1/2^{10})$ = 0.123

37. Find the MGF of a uniform distribution in (a, b)?

Ans :

$$M_X(t) = \frac{1}{b-a}\int_a^b e^{tx}dx \qquad = \frac{e^{bt} - e^{at}}{(b-a)t}$$

38. Find the MGF of a RV X which is uniformly distributed over (-2, 3)

$$M_X(t) = \frac{1}{5}\int_{-2}^{3} e^{tx}dx = \frac{e^{3t} - e^{-2t}}{5t} \ for \ t \neq 0$$

39. The M.G.F of a R.V X is of the form M_x (t) = $(0.4e^t + 0.6)^8$ what is the M.G.F of the R.V, Y = 3X +2

$M_Y(t) = e^{2t}M_X(t) = e^{2t}((0.4)e^{3t} + 0.6)^8$

40. If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$ find P(X<0)

Ans : Let X follows uniform distribution in (a,b)

mean = $\frac{b+a}{2} = 1$ and Variance = $\frac{(b-a)^2}{12} = \frac{4}{3}$

$\therefore$ a+b = 2 $(b-a)^2 = 16 \Rightarrow b - a = \pm 4$

Solving we get a=-1 b=3

$\therefore$ f(X) = $\frac{1}{4}$, - 1 < x < 3

$\therefore$ P[X< 0] = $\int_{-1}^{0} f(x)dx = \frac{1}{4}$

41. A RV X has a uniform distribution over (-4, 4) compute $P(|X|> 2)$

Ans :

$$f(x) = \begin{cases} \frac{1}{8}, & -4 < x < 4 \\ 0 \text{ other wise} \end{cases}$$

$$P(|X|> 2) = 1 - P(|X| \le 2) = 1 - P(-2 < X < 2) = 1 - \int_{-2}^{2} f(x)dx = 1 - \frac{4}{8} = \frac{1}{2}$$

42. If X is Uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.Find the p.d.f of Y=tan X.

Sol: $f_X(x) = \frac{1}{\pi}$; $X = \tan^{-1} Y \Rightarrow \frac{dx}{dy} = \frac{1}{1+y^2}$.

$$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \Rightarrow f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty$$

43. If X is uniformly distributed in (-1,1).Find the p.d.f of $y = \sin\frac{\pi x}{2}$.

Sol:

$$f_X(x) = \begin{cases} \frac{1}{2}, -1 < x < 1 \\ 0, otherwise \end{cases},$$

$$x = \frac{2\sin^{-1} y}{\pi} \Rightarrow \frac{dx}{dy} = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} \quad \text{for} \quad -1 \le y \le 1$$

$$f_Y(y) = \frac{1}{2}\left[\frac{2}{\pi}\frac{1}{\sqrt{1-y^2}}\right] \Rightarrow f_Y(y) = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}} \quad \text{for} \quad -1 \le y \le 1$$

44.The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$ what is the probability that a repair takes at least 10 hours given that its duration exceeds 9 hours?

Ans :

Let X be the RV which represents the time to repair machine.

P[X≥ 10/ X ≥ 9] = P [X≥ 1] (by memory less property)

$$= \int_{1}^{\infty} \frac{1}{2} e^{-x/2} dx = 0.6065$$

45. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{3}$ what is the probability that the repair time exceeds 3 hours?

Ans : X – represents the time to repair the machine

$$\therefore f(x) = \frac{1}{3}e^{-x/3} > 0$$

$$P(X > 3) = \int_3^\infty \frac{1}{3}e^{-x/3}dx = e^{-1} = 0.3679$$

46.Find the M.G.F of an exponential distribution with parameter λ.

Sol: $M_x(t) = \lambda\int_0^\infty e^{tx}e^{-\lambda x}dx = \lambda\int_0^\infty e^{-(\lambda-t)x}dx = \frac{\lambda}{\lambda - t}$

47. State the memory less property of the exponential distribution.

Soln: If X is exponential distributed with parameter λ then

$P(X > s + t / X > s) = P(X > t)$ for s , t >0

48. If X has a exponential distribution with parameter λ,find the p.d.f of Y=log X.

Sol: Y=log X$\Rightarrow e^y = x \Rightarrow \frac{dx}{dy} = e^y$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|. \Rightarrow f_Y(y) = e^y \lambda e^{-\lambda e^y}.$$

49. If X has Exponential Distribution with parameter 1,find the p.d.f of $Y = \sqrt{X}$.

Sol: $Y = \sqrt{X} \Rightarrow X = Y^2 \quad f_X(x) = e^{-x}, x > 0.$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 2ye^{-x} = 2ye^{-y^2}, y > 0.$$

50 . Write the M.G.F of Gamma distribution

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \frac{\lambda^\gamma}{\Gamma\gamma}\int_0^\infty e^{-(\lambda-t)x}x^{\gamma-1}dx \qquad = \qquad \frac{\lambda^\gamma}{\Gamma\gamma}\frac{\Gamma\gamma}{(\lambda-t)^\gamma}$$

$$\therefore Mx(t) = \left(1 - \frac{t}{\lambda}\right)^{-\gamma}$$

51. Define Normal distribution

A normal distribution is a continuous distribution given by

$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$ where X is a continuous normal variate distributed with density function $f(X) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2}$ with mean μ and standard deviation σ

52. What are the properties of Normal distribution

(i) The normal curve is symmetrical when p = q or p $\approx$ q.

(ii) The normal curve is single peaked curve.

(iii) The normal curve is asymptotic to x-axis as y decreases rapidly when x increases numerically.

(iv) The mean, median and mode coincide and lower and upper quartile are equidistant from the median.

(v) The curve is completely specified by mean and standard deviation along with the value y_o.

53. Write any four properties of normal distribution.

Sol: (1) The curve is Bell shaped

(2) Mean, Median, Mode coincide

(3) All odd moments vanish

(4) x - axis is an asymptote of the normal curve

54. If X is a Normal variate with mean30 and SD 5.Find P [26<X<40].

Sol: P [26<X<40] =P $[-0.8 \le Z \le 2]$ where $Z = \dfrac{X-30}{5}$ $\left\{\because Z = \dfrac{X-\mu}{\sigma}\right\}$

$= P[0 \le Z \le 0.8] + P[0 \le Z \le 2]$

= 0.2881+0.4772

=0.7653.

55. If X is normally distributed RV with mean 12 and SD 4.Find P [X$\le$ 20].

Sol: $P[X \le 20] = P[Z \le 2]$ where $Z = \dfrac{X-12}{4}$ $\left\{\because Z = \dfrac{X-\mu}{\sigma}\right\}$

$= P[-\infty \le Z \le 0] + P[0 \le Z \le 2]$

=0.5+0.4772

=0.9772.

56. **If X is a N(2,3),find** $P\left[Y \ge \dfrac{3}{2}\right]$ **where Y+1=X.**

Answer:

$$P\left[Y \ge \frac{3}{2}\right] = P\left[X - 1 \ge \frac{3}{2}\right] = P[X \ge 2.5] = P[Z \ge 0.17] = 0.5 - P[0 \le Z \le 0.17]$$

$$= 0.5 - 0.0675 = 0.4325$$

57.. If X is a RV with p.d.f $f(x) = \dfrac{x}{12}$ **in 1 < x < 5 and =0, otherwise.Find the p.d.f of Y=2X-3.**

Sol: Y=2X-3 $\Rightarrow \dfrac{dx}{dy} = \dfrac{1}{2}$

$f_Y(y) = f_X(x)\left|\dfrac{dx}{dy}\right| = \dfrac{y+3}{4}$,. in -1<y<7.

58.. If X is a Normal R.V with mean zero and variance σ^2 **Find the p.d.f of** $Y = e^X$ **.**

Sol: $Y = e^X \Rightarrow \log Y = X \Rightarrow \dfrac{dx}{dy} = \dfrac{1}{y}$

$f_Y(y) = f_X(x)\left|\dfrac{dx}{dy}\right| = \dfrac{1}{y} f_X(\log y)$

$= \dfrac{1}{\sigma y\sqrt{2\pi}} \exp\left(-\left(\log y - \mu^2\right)/2\sigma^2\right)$

59. If X has the p.d.f $f(x) = \begin{cases} x, 0 < x < 1 \\ 0, \; otherwise \end{cases}$ **find the p.d.f of** $Y = 8X^3$.

Sol: $Y = 8X^3 \Rightarrow X = \frac{1}{2}Y^{\frac{1}{3}} \quad \Rightarrow \frac{dx}{dy} = \frac{1}{6}y^{-\frac{2}{3}}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = (x)\left(\frac{1}{6}Y^{-\frac{2}{3}}\right) = \frac{1}{12}Y^{-\frac{1}{3}}, Y > 0$$

60. The p.d.f of a R.V X is $f(x) = 2x, 0 < x < 1$. **Find the p.d.f of** $Y = 3X + 1$.

Sol: $Y = 3X + 1 \Rightarrow X = \frac{Y-1}{3}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = 2x\frac{1}{3} = 2\left(\frac{y-1}{3}\right)\left(\frac{1}{3}\right) = \frac{2(y-1)}{9}.,1 < y < 4.$$

Moment generating functions

1. Define n^{th} Moments about Origin

The n^{th} moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

For discrete RV X, the n^{th} moment is defined as $E(X^n) = \sum_i x_i^n p_i = \mu_n', n \geq 1$

For continuous RV X, the n^{th} moment is defined as $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \mu_n', n \geq 1$

2.Define n^{th} Moments about Mean

The n^{th} central moment of a discrete RV X is its moment about its mean $\overline{X}$ and is defined as

$$E\left(X - \overline{X}\right)^n = \sum_i \left(x_i - \overline{X}\right)^n p_i = \mu_n, n \geq 1$$

The n^{th} central moment of a continuous RV X is defined as

$$E\left(X - \overline{X}\right)^n = \int_{-\infty}^{\infty} \left(x - \overline{X}\right)^n f(x)dx = \mu_n, n \geq 1$$

3..Define Variance

The second moment about the mean is called variance and is represented as $\sigma_x^{\ 2}$

$$\sigma_x^{\ 2} = E\left[X^2\right] - \left[E(X)\right]^2 = \mu_2' - \left(\mu_1'\right)^2$$

The positive square root σ_x of the variance is called the standard deviation.

4.Define Moment Generating Functions (M.G.F)

Moment generating function of a RV X about the origin is defined as

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx} P(x), if X is discrete. \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, if X is continous. \end{cases}$$

Moment generating function of a RV X about the mean is defined as

$$M_{X-\mu}(t) = E(e^{t(x-\mu)})$$

5. Properties of MGF

1. $M_{X=a}(t) = e^{-at} M_X(t)$

Proof:

$M_{X=a}(t) = E(e^{t(x-a)}) = E(e^{tx}.e^{-at}) = E(e^{tx})e^{-at} = e^{-at} M_X(t)$

2. If X and Y are two independent RVs, then $M_{X+Y}(t) = M_X(t).M_Y(t)$

Proof:

$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX+tY}) = E(e^{tX}.e^{tY}) = E(e^{tX}).E(e^{tY}) = M_X(t).M_Y(t)$

3. $If M_X(t) = E(e^{tx}) then M_{cX}(t) = M_X(ct)$

Proof:

$M_{cX}(t) = E(e^{tcX}) = E(e^{(ct)X}) = M_X(ct)$

4. If Y=aX+b then $M_Y(t) = e^{bt} M_X(at)$ where $M_X(t)$=MGF of X.

5. If $M_{X_1}(t) = M_{X_2}(t)$ for all t, then $F_{X_1}(x) = F_{X_2}(x)$ for all x.

UNIT-I RANDOM VARIABLE

1. If the RV X takes the values 1, 2, 3 and 4 such that $2P(X=1)=3P(X=2)=P(X=3)=5P(X=4)$, find the probability distribution and cumulative distribution function of X.

2. A RV X has the following probability distribution.

X:	-2	-1	0	1	2	3
P(x):	0.1	K	0.2	2K	0.3	3K

Find (1) K, (2) $P(X<2)$, $P(-2< X <2)$, (3) CDF of X, (4) Mean of X.

3. If X is RV with probability distribution

X:	1	2	3
P(X):	1/6	1/3	1/2

Find its mean and variance and $E(4X^3 +3X+11)$.

4. A RV X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
P(x):	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find (1) K, (2) $P(X<2)$, $P(1.5 <X< 4.5/X>2)$, (3) The smallest value of λ for which $P(X \le \lambda) > 1/2$.

5. A RV X has the following probability distribution.

X:	0	1	2	3	4
P(x):	K	3K	5K	7K	9K

Find (1) K, (2) $P(X<3)$ and $P(0<X< 4)$, (3) Find the distribution function of X.

6. If the density function of a continuous RV X is given by $f(x)=\begin{cases} ax, & 0\le x\le 1 \\ a, & 1\le x\le 2 \\ 3a-ax, & 2\le x\le 3 \\ 0, & Otherwise \end{cases}$

Find i)a ii)CDF of X.

7. A continuous RV X that can assume any value between x=2 and x=5 has a density function given by $f(x)=k(1+x)$. Fnd $P(X<4)$.

8. If the density function of a continuous RV X is given by $f(x)=kx^2e^{-x}, x>0$. Find k, mean and variance.

9. If the cdf of a continuous RV X is given by $F(x)=\begin{cases} 0, & x<0 \\ x^2, & 0\le x<\frac{1}{2} \\ 1-\frac{3}{25}(3-x^2), & \frac{1}{2}\le x<3 \\ 1, & x\ge 3 \end{cases}$

Find the pdf of X and evaluate $P(|X|\le 1)$ and $P(\frac{1}{3}\le X<4)$.

10. A continuous RV X has the pdf $f(x)=Kx^2e^{-x}, x\ge 0$. Find the r^{th} moment about

origin.Hence find mean and variance of X.

11. Find the mean, variance and moment generating function of a binomialdistribution.

12. 6 dice are thrown 729 times.How many times do you expect atleast three dice to show 5or 6?

.13. It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the no. of packets containing atleast, exactly and atmost 2 defective items in a consignment of 1000 packets using (i) Binomial distribution

14. Find mean , variance and MGF of Geometric distribution.

15. The pdf of the length of the time that a person speaks over phone is

$$f(x) = \begin{cases} Be^{\frac{-x}{6}}, & x > 0 \\ 0, & otherwise \end{cases}$$

what is the probability that the person wil talk for (i) more than 8 minutes (ii)less than 4 minutes (iii) between 4 and 8 minutes.

16. State and prove the memory less property of the exponential distribution.

17. If the service life, in hours, of a semiconductor is a RV having a Weibull distribution with the parameters α = 0.025 and β = 0.5,
    1. How long can such a semiconductor be expected to last?
    2. What is the probability that such a semiconductor will still be in operating condition after 4000 hours?

Unit II Two Dimensional Random Variables

1.Define Two-dimensional Random variables.

Let S be the sample space associated with a random experiment E. Let X=X(S) and Y=Y(S) be two functions each assigning a real number to each $s \in S$. Then (X,Y) is called a two dimensional random variable.

2. The following table gives the joint probability distribution of X and Y. Find the mariginal density functions of X and Y.

Y /X	**1**	**2**	**3**
1	**0.1**	**0.1**	**0.2**
2	**0.2**	**0.3**	**0.1**

Answer:

The marginal density of X

$$P(X = x_i) = p_{i*} = \sum_j p_{ij}$$

X	1	2	3
P(X)	0.3	0.4	0.3

The marginal density of Y

$$P(Y = y_j) = p_{*j} = \sum_i p_{ij}$$

Y	1	2
P(Y)	0.4	0.6

3.If $f(x,y) = kxye^{-(x^2+y^2)}, \ x \geq 0, y \geq 0$ **is the joint pdf, find** k.

Answer:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1 \Rightarrow \int_0^{\infty}\int_0^{\infty} kxye^{-(x^2+y^2)}dydx = 1$$

$$k\int_0^{\infty} xe^{-x^2}dx\int_0^{\infty} ye^{-y^2}dx = 1 \Rightarrow \frac{k}{4} = 1$$

$$\therefore k = 4$$

4. Let the joint pdf of X and Y is given by $f(x,y) = \begin{cases} cx(1-x), 0 \leq x \leq y \leq 1 \\ 0 \qquad ,otherwise \end{cases}$

Find the value of C.

Answer:

$$\int_0^1\int_0^y Cx(1-x)dxdy = 1 \Rightarrow \frac{C}{6}\int_0^1 (3y^2 - 2y^3)dy = 1 \Rightarrow \frac{C}{6}\left[1 - \frac{1}{2}\right] = 1$$

5. The joint p.m.f of (X,Y) is given by $P(x,y) = k(2x+3y), \ x = 0,1,2; \ y = 1,2,3.$**.Find the marginal probability distribution of X.**

Answer:

X \Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

$$\sum_y\sum_x P(x,y) = 1 \Rightarrow 72k = 1 \therefore k = \frac{1}{72}$$

Marginal distribution of X:

X	0	1	2
P(X)	18/72	24/72	30/72

6. If X and Y are independent RVs with variances 8and 5.find the variance of 3X+4Y.

Answer:

Given Var(X)=8 and Var(Y)=5

To find:var(3X-4Y)

We know that Var(aX - bY)=a^2Var(X)+b^2Var(Y)

var(3X- 4Y)=3^2Var(X)+4^2Var(Y) =(9)(8)+(16)(5)=152

7. Find the value of k if $f(x,y) = k(1-x)(1-y)$ for $0 < x, y < 1$ is to be joint density function.

Answer:

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$

$$\int_0^1\int_0^1 k(1-x)(1-y)dxdy = 1 \Rightarrow k\left[\int_0^1 (1-x)dx\right]\left[\int_0^1 (1-y)dy\right] = 1$$

$$k\left[x - \frac{x^2}{2}\right]_0^1\left[y - \frac{y^2}{2}\right]_0^1 = 1 \Rightarrow \frac{k}{4} = 1 \qquad \therefore k = 4$$

8. If X and Y are random variables having the joint p.d.f

$f(x,y) = \frac{1}{8}(6-x-y),\ \ 0 < x < 2,\ 2 < y < 4$, **find P(X<1,Y<3)**

Answer:

$$P(X < 1, Y < 3) = \frac{1}{8}\int_0^1\int_2^3 (6-x-y)dydx = \frac{1}{8}\int_0^1\left(\frac{7}{2} - x\right)dx = \frac{3}{8}$$

9. The joint p.d.f of (X,Y)is given by $f(x,y) = \frac{1}{4}(1+xy), |x| < 1, |y| < 1$ and $= 0$,otherwise.

Show that X and Y are not independent.

Answer:

Marginal p.d.f of X :

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-1}^{1}\frac{1}{4}(1+xy)dy = \frac{1}{2}, \quad -1 < x < 1$$

$$f(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

Marginal p.d.f of Y :

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-1}^{1}\frac{1}{4}(1+xy)dx = \frac{1}{2}, \quad -1 < y < 1$$

$$f(y) = \begin{cases} \frac{1}{2}, & -1 < y < 1 \\ 0, & otherwise \end{cases}$$

Since $f(x)f(y) \neq f(x,y)$,X and Y are not independent.

10.The conditional p.d.f of X and Y=y is given by $f\left(\frac{x}{y}\right) = \frac{x+y}{1+y}e^{-x}, 0 < x < \infty, 0 < y < \infty,$

find $P[X < 1/Y = 2]$.

Answer:

When y =2, $f(x/y=2) = \frac{x+2}{3}e^{-x}$

$$\therefore P[X < 1/Y = 2] = \int_0^1 \frac{x+2}{3}e^{-x}dx = \frac{1}{3}\int_0^1 xe^{-x}dx + \frac{2}{3}\int_0^1 e^{-x}dx = 1 - \frac{4}{3}e^{-1}$$

11. The joint p.d.f of two random variables X and Y is given by

$f(x,y)=\frac{1}{8}x(x-y), 0<x<2, -x<y<x$ **and** $=0$**, otherwise.**

Find $f(y/x)$

Answer:

$$f(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{-x}^{x}\frac{1}{8}x(x-y)dy=\frac{x^3}{4},\quad 0<x<2$$

$$f(y/x)=\frac{f(x,y)}{f(x)}=\frac{x-y}{2x^2}, -x<y<x$$

12. If the joint pdf of (X,Y) is $f(x,y)=\frac{1}{4},\ 0\le x,\ y<2$ **,find** $P[X+Y\le 1]$

Answer:

$$P[X+Y\le 1]=\int_0^1\int_0^{1-y}\frac{1}{4}dxdy=\frac{1}{4}\int_0^1(1-y)dy=\frac{1}{8}.$$

13. If the joint pdf of (X,Y) is $f(x,y)=6e^{-2x-3y},\ x\ge 0, y\ge 0,$ **find the conditional density of Y given X.**

Answer:

Given $f(x,y)=6e^{-2x-3y},\ x\ge 0, y\ge 0,$

The Marginal p.d.f of X:

$$f(x)=\int_0^{\infty}6e^{-2x-3y}dy=2e^{-2x},\ x\ge 0$$

Conditional density of Y given X:

$$f(y/x)=\frac{f(x,y)}{f(x)}=\frac{6e^{-2x-3y}}{2e^{-2x}}=3e^{-3y}, y\ge 0.$$

14.**Find the probability distribution of (X+Y) given the bivariate distribution of (X,Y).**

X \Y	**1**	**2**
1	**0.1**	**0.2**
2	**0.3**	**0.4**

Answer:

X+Y	P(X+Y)
2	P(2)=P(X=1,Y=1)=0.1
3	P(3)=P(X=1,Y=2)+P(X=2,Y=1)=0.2+0.3=0.5
4	P(4)=P(X=2,Y=2)=0.4

$\therefore$

X+Y	2	3	4
Probability	0.1	0.5	0.4

15. The joint p.d.f of (X,Y) is given by $f(x,y)=6e^{-(x+y)},\ 0\le x,y\le\infty.$ **Are X and Y independent?**

Answer:

Marginal density of X:

$$f(x)=\int_{-\infty}^{\infty}f(x,y)dy=\int_0^{\infty}6e^{-(x+y)}dy=e^{-x},\ 0\le x$$

Marginal density of Y;

$$f(y)=\int_{-\infty}^{\infty}f(x,y)dx=\int_0^{\infty}6e^{-(x+y)}dx=e^{-y},\ y\le\infty$$

$$\Rightarrow f(x)f(y)=f(x,y)$$

$\therefore$ X and Y are independent.

16. The joint p.d.f of a bivariate R.V (X,Y) is given by

$$f(x,y)=\begin{cases}4xy, & 0<x<1, y<1\infty \\ 0 & ,otherwise\end{cases}$$ **.Find p(X+Y<1)**

Answer:

$$P[X+Y<1]=\int_0^1\int_0^{1-y} 4xydxdy = 2\int_0^1 y(1-y)^2dy$$

$$=2\left[\frac{y^2}{2}-\frac{2y^3}{3}+\frac{y^4}{4}\right]_0^1$$

$$=2\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]=\frac{1}{6}$$

17. Define Co – Variance:

If X and Y are two two r.v.s then co – variance between them is defined as

Cov (X, Y) = E {X – E (X)} {Y – E (Y)}

(ie) Cov (X, Y) = E (XY) – E(X) E (Y)

18. State the properties of Co – variance;

1. If X and Y are two independent variables, then Cov (X,Y) = 0. But the Converse need not be true
2. Cov (aX, bY) = ab Cov (X,Y)
3. Cov (X + a,Y +b) = Cov (X,Y)
4. $Cov\left(\frac{X-\overline{X}}{\sigma_X},\frac{Y-\overline{Y}}{\sigma_Y}\right)=\frac{1}{\sigma_X\sigma_Y}Cov(X,Y)$
5. Cov(aX+b,cY+d)=acCov(X,Y)
6. Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z)
7. $Cov(aX+bY,cX+dY)=ac\sigma_X{}^2+bd\sigma_Y{}^2+(ad+bc)Cov(X,Y)$

   $where\,\sigma_X{}^2=Cov(X,X)=\text{var}(X)\,and\,\sigma_Y{}^2=Cov(Y,Y)=\text{var}(Y)$

19.Show that Cov(aX+b,cY+d)=acCov(X,Y)

Answer:

Take U= aX+b and V= cY+d

Then E(U)=aE(X)+b and E(V)= cE(Y)+d

U-E(U)= a[X-E(X)] and V-E(V)=c[Y-E(Y)]

Cov(aX+b,cY+d)= Cov(U,V)=E[{U-E(U)}{V-E(V)}] = E[a{X-E(X)} c{Y-E(Y)}]

=ac E[{X-E(X)}{Y-E(Y)}]=acCov(X,Y)

20.If X&Y are independent R.V's ,what are the values of Var(X_1+X_2)and Var(X_1-X_2)

Answer:

$\text{Var}(X_1\pm X_2)=\text{Var}(X_1)+\text{Var}(X_2)$ (Since X andY are independent RV then

$\text{Var}(aX\pm bX)=a^2\text{Var}(X)+b^2\text{Var}(X)$)

21. If Y_1& Y_2 are independent R.V's ,then covariance (Y_1,Y_2)=0.Is the converse of the above statement true?Justify your answer.

Answer:

The converse is not true . Consider

$\text{X - N(0,1)and Y} = X^2 \quad \sin ceX-N(0,1),$

$E(X)=0;\ E(X^3)=E(XY)=0\sin ce\ \ all\ odd\ moments\quad vanish.$

$\therefore \text{cov}(XY)=E(XY)-E(X)E(Y)=E(X^3)-E(X)E(Y)=0$

$\therefore \text{cov}(XY)=0\quad but\quad X\,\&\,Y\quad are\,independent$

22. Show that $\text{cov}^2(X,Y)\le\text{var}(X)\,\text{var}(Y)$

Answer:

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y)$$

We know that $[E(XY)]^2 \le E(X^2)E(Y^2)$

$$\operatorname{cov}^2(X,Y) = [E(XY)]^2 + [E(X)]^2[E(Y)]^2 - 2E(XY)E(X)E(Y)$$
$$\le E(X)^2E(Y)^2 + [E(X)]^2[E(Y)]^2 - 2E(XY)E(X)E(Y)$$
$$\le E(X)^2E(Y)^2 + [E(X)]^2[E(Y)]^2 - E(X^2)E(Y)^2 - E(Y^2)E(X)^2$$
$$= \{E(X^2) - [E(X)]^2\}\{E(Y^2) - [E(Y)]^2\} \le \operatorname{var}(X)\operatorname{var}(Y)$$

23.If X and Y are independent random variable find covariance between X+Y and X-Y.

Answer:

$$\operatorname{cov}[X+Y, X-Y] = E[(X+Y)(X-Y)] - [E(X+Y)E(X-Y)]$$
$$= E[X^2] - E[Y^2] - [E(X)]^2 + [E(Y)]^2$$
$$= \operatorname{var}(X) - \operatorname{var}(Y)$$

24. X and Y are independent random variables with variances 2 and 3.Find the variance 3X+4Y.

Answer:

Given var(X) = 2, var(Y) = 3

We know that var(aX+Y) = a^2var(X) + var(Y)

And var(aX+bY) = a^2var(X) + b^2var(Y)

var(3X+4Y) = 3^2var(X) + 4^2var(Y) = 9(2) + 16(3) = 66

25. Define correlation

The correlation between two RVs X and Y is defined as

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(xy)\,dxdy$$

26. Define uncorrelated

Two RVs are uncorrelated with each other, if the correlation between X and Y is equal to the product of their means. i.e.,E[XY]=E[X].E[Y]

27. If the joint pdf of (X,Y) is given by $f(x,y) = e^{-(x+y)}, x \ge 0, y \ge 0.$ find E(XY).

Answer:

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dxdy = \int_0^{\infty}\int_0^{\infty} xye^{-(x+y)}\,dxdy = \int_0^{\infty} xe^{-x}dx\int_0^{\infty} ye^{-y}dy = 1$$

28. A R.V X is uniformly distributed over(-1,1) and Y=X^2. Check if X and Y are correlated?

Answer:

Given X is uniformly distributed in (-1,1),pdf of X is $f(x) = \frac{1}{b-a} = \frac{1}{2},\ -1 \le x \le 1$

$$E(X) = \frac{1}{2}\int_{-1}^{1} x\,dx = 0 \text{ and } E(XY) = E(X^3) = 0$$

$$\therefore \operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0 \qquad \Rightarrow r(X,Y) = 0$$

Hence X and Y are uncorrelated.

29. X and Y are discrete random variables. If $\operatorname{var}(X) = \operatorname{var}(Y) = \sigma^2$,

$\operatorname{cov}(X,Y) = \frac{\sigma^2}{2}$, **find** $\operatorname{var}(2X - 3Y)$

Answer:

$$\operatorname{var}(2X-3Y)=4\operatorname{var}(X)+9\operatorname{var}(Y)-12\operatorname{cov}(X,Y)$$

$$=13\sigma^2-12\frac{\sigma^2}{2}=7\sigma^2$$

30. If $\operatorname{var}(X)=\operatorname{var}(Y)=\sigma^2$, $\operatorname{cov}(X,Y)=\frac{\sigma^2}{2}$, **find the correlation between** $2X+3$ **and** $2Y-3$

Answer:

$$r(aX+b,cY+d)=\frac{ac}{|ac|}r(X,Y)\ \textit{where}\ a\neq 0,c\neq 0$$

$$\therefore r(2X+3,2Y-3)=\frac{4}{|4|}r(X,Y)=r(X,Y)=\frac{\operatorname{cov}(X,Y)}{\sigma_X\sigma_Y}=\frac{\sigma^2/2}{\sigma.\sigma}=\frac{1}{2}$$

31. Two independent random variables X and Y have 36 and 16.Find the correlation co-efficient between X+Y and X-Y

Answer:

$$\therefore r(X+Y,X-Y)=\frac{\sigma_X{}^2-\sigma_Y{}^2}{\sigma_X{}^2+\sigma_Y{}^2}=\frac{36-16}{36+16}=\frac{20}{52}=\frac{4}{13}$$

32. If the lines of regression of Y on X and X on Y are respectively $a_1X+b_1Y+c_1=0$ **and** $a_2X+b_2Y+c_2=0$ **,prove that** $a_1b_2\leq a_2b_1$ **.**

Answer:

$$b_{yx}=-\frac{a_1}{b_1}\qquad \textit{and}\qquad b_{xy}=-\frac{b_2}{a_2}$$

Since $r^2=b_{yx}b_{yx}\leq 1\quad\Rightarrow\frac{a_1}{b_1}.\frac{b_2}{a_2}\leq 1$

$$\therefore a_1b_2\leq a_2b_1$$

33. State the equations of the two regression lines. what is the angle between them?

Answer:

Regression lines:

$$y-\bar{y}=r\frac{\sigma y}{\sigma x}(x-\bar{x})\ \ \textit{and}\ \ x-\bar{x}=r\frac{\sigma x}{\sigma y}(y-\bar{y})$$

$$\text{Angle }\theta=\tan^{-1}\left[\frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x{}^2+\sigma_y{}^2}\right)\right]$$

34. The regression lines between two random variables X and Y is given by $3X+Y=10$ **and** $3X+4Y=12$ **.Find the correlation between X and Y.**

Answer:

$$3X+4Y=12\qquad\Rightarrow b_{yx}=-\frac{3}{4}$$

$$3X+Y=10\qquad\Rightarrow b_{xy}=-\frac{1}{3}$$

$$r^2=\left(-\frac{3}{4}\right)\left(-\frac{1}{3}\right)=\frac{1}{4}\qquad\Rightarrow r=-\frac{1}{2}$$

35. Distinguish between correlation and regression.

Answer:

By correlation we mean the casual relationship between two or more variables. By regression we mean the average relationship between two or more variables.

36. State the Central Limit Theorem.

Answer:

If $x_1, x_2, \ldots\ldots\ldots, x_n$ are n independent identically distributed RVs with men μ and S.D σ and if $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, then the variate $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ has a distribution that approaches the standard normal distribution as $n \to \infty$ provided the m.g.f of x_i exist.

37. The lifetime of a certain kind of electric bulb may be considered as a RV with mean 1200 hours and S.D 250 hours. Find the probability that the average life time of exceeds 1250 hours using central limit theorem.

Solution:

Let X denote the life time of the 60 bulbs.

Then μ = E(X)= 1200 hrs. and Var(X)=(S.D)^2= σ^2=(250)^{2}hrs.

Let $\bar{X}$ denote the average life time of 60 bulbs.

By Central Limit Theorem, $\bar{X}$ follows $N\left(\mu, \frac{\sigma^2}{n}\right)$.

Let $Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$ be the standard normal variable

$$P[\bar{X} > 1250] = P[Z > 1.55]$$
$$= 0.5 - P[0 < Z < 1.55]$$
$$= 0.5 - 0.4394 = 0.0606$$

38.Joint probability distribution of (X, Y)

Let (X,Y) be a two dimensional discrete random variable. Let P(X=x_i,Y=y_j)=p_{ij}. p_{ij} is called the probability function of (X,Y) or joint probability distribution. If the following conditions are satisfied

1.$p_{ij} \ge 0$ for all i and j

2.$\sum_j \sum_i p_{ij} = 1$

The set of triples (x_i,y_j, p_{ij}) i=1,2,3……. And j=1,2,3……is called the Joint probability distribution of (X,Y)

39.Joint probability density function

If (X,Y) is a two-dimensional continuous RV such that

$$P\left\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2} and y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right\} = f(x, y)dxdy$$

Then f(x,y) is called the joint pdf of (X,Y) provided the following conditions satisfied.

1. $f(x, y) \ge 0 \, for \; all \; (x, y) \in (-\infty, \infty)$

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1 \; and \; f(x, y) \ge 0 \; for \, all (x, y) \in (-\infty, \infty)$

40.Joint cumulative distribution function (joint cdf)

If (X,Y) is a two dimensional RV then $F(x, y) = P(X \le x, Y \le y)$ is called joint cdf of (X,Y)

In the discrete case,

$$F(x, y) = \sum_{\substack{j \\ y_j \le y}} \sum_{\substack{i \\ x_i \le x}} p_{ij}$$

In the continuous case,

$$F(x, y) = P(-\infty < X \le x, -\infty < Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x, y)dxdy$$

41.Marginal probability distribution(Discrete case)

Let (X,Y) be a two dimensional discrete RV and $p_{ij}=P(X=x_i,Y=y_j)$ then

$$P(X = x_i) = p_{i*} = \sum_j p_{ij}$$

is called the Marginal probability function.

The collection of pairs $\{x_i,p_{i*}\}$ is called the Marginal probability distribution of X.

If $P(Y = y_j) = p_{*j} = \sum_i p_{ij}$ then the collection of pairs $\{x_i,p_{*j}\}$ is called the Marginal probability distribution of Y.

42.Marginal density function (Continuous case)

Let f(x,y) be the joint pdf of a continuous two dimensional RV(X,Y).The marginal density function of X is defined by $f(x) = \int_{-\infty}^{\infty} f(x,y)dy$

The marginal density function of Y is defined by $f(y) = \int_{-\infty}^{\infty} f(x,y)dx$

43.Conditional probability function

If $p_{ij}=P(X=x_i,Y=y_j)$ is the Joint probability function of a two dimensional discrete RV(X,Y) then the conditional probability function X given $Y=y_j$ is defined by

$$P\left[X = x_i / Y = y_j\right] = \frac{P\left[X = x_i \cap Y = y_j\right]}{P\left[Y = y_j\right]}$$

The conditional probability function Y given $X=x_i$ is defined by

$$P\left[Y = y_j / X = x_i\right] = \frac{P\left[X = x_i \cap Y = y_j\right]}{P\left[X = x_i\right]}$$

44.Conditional density function

Let f(x,y) be the joint pdf of a continuous two dimensional RV(X,Y).Then the Conditional density function of X given Y=y is defined by $f(X/Y) = \frac{f_{XY}(x,y)}{f_Y(y)}$,where f(y)=marginal p.d.f of Y.

The Conditional density function of Y given X=x is defined by $f(Y/X) = \frac{f_{XY}(x,y)}{f_X(x)}$, where f(x)=marginal p.d.f of X.

45.Define statistical properties

Two jointly distributed RVs X and Y are statistical independent of each other if and only if the joint probability density function equals the product of the two marginal probability density function

i.e., f(x,y)=f(x).f(y)

46.The joint p.d.f of (X,Y) is given by $f(x,y) = e^{-(x+y)}$ $0 \le x, y \le \infty$ .Are X and Y are independent?

Answer:

Marginal densities:

f(x)= $\int_0^{\infty} e^{-(x+y)}dy = e^{-x}$ and f(y)= $\int_0^{\infty} e^{-(x+y)}dx = e^{-y}$

X and Y are independent since f(x,y)=f(x).f(y)

47.Define Co – Variance :

If X and Y are two two r.v.s then co – variance between them is defined as

Cov (X, Y) = E {X – E (X)} {Y – E (Y)}

(ie) Cov (X, Y) = E (XY) – E(X) E (Y)

48. State the properties of Co – variance;

1. If X and Y are two independent variables, then Cov (X,Y) = 0. But the Converse need not be true
2. Cov (aX, bY) = ab Cov (X,Y)
3. Cov (X + a,Y +b) = Cov (X,Y)
4. $Cov\left(\frac{X-\overline{X}}{\sigma_X},\frac{Y-\overline{Y}}{\sigma_Y}\right)=\frac{1}{\sigma_X\sigma_Y}Cov(X,Y)$
5. Cov(aX+b,cY+d)=acCov(X,Y)
6. Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z)
7. $Cov(aX+bY,cX+dY)=ac\sigma_X{}^2+bd\sigma_Y{}^2+(ad+bc)Cov(X,Y)$

$$where\,\sigma_X{}^2=Cov(X,X)=\text{var}(X)\,and\,\sigma_Y{}^2=Cov(Y,Y)=\text{var}(Y)$$

49.Show that Cov(aX+b,cY+d)=acCov(X,Y)

Answer:

Take U= aX+b and V= cY+d

Then E(U)=aE(X)+b and E(V)= cE(Y)+d

U-E(U)= a[X-E(X)] and V-E(V)=c[Y-E(Y)]

Cov(aX+b,cY+d)= Cov(U,V)=E[{U-E(U)}{V-E(V)}] = E[a{X-E(X)} c{Y-E(Y)}]

=ac E[{X-E(X)}{Y-E(Y)}]=acCov(X,Y)